ISRAEL JOURNAL OF MATHEMATICS **152** (2006), 29–47

PROPER FORCING AND RECTANGULAR RAMSEY THEOREMS

BY

JINDŘICH ZAPLETAL*

Department of Mathematics, Little Hall 358
University of Florida, Gainesville, FL 32611, USA
e-mail: zapletal@math.ufl.edu

ABSTRACT

I prove forcing preservation theorems for products of definable partial orders preserving cofinality of the meager or null ideal. Rectangular Ramsey theorems for the related ideals follow from the proofs.

1. Introduction

The following fact is the archetype of a rectangular Ramsey theorem.

FACT 1.1 ([4] 19.6): For every Borel coloring $\mathbb{R}^2 = \bigcup_n A_n$ of the plane there is a monochromatic rectangle with perfect sides.

Many variations on it have been proved, for higher dimensions as well as with other demands on the size of the sides of the monochromatic rectangle. In this paper I will present a theorem which yields many generalizations of the above fact, connects the rectangular Ramsey theory with the theory of proper forcing and determinacy, and isolates possible new open questions.

In order to facilitate the statement of the theorem and the surrounding discussion, it is convenient to introduce the following notation.

* Partially supported by grant GA ČR 201-03-0933, NSF grants DMS 0071437 and DMS 0300201, and a visiting appointment at CRM, Universitat Autònoma de Barcelona.
Received January 25, 2004 and in revised form September 30, 2004

Definition 1.2: Suppose that $\vec{I}$ is a finite or countable sequence of σ-ideals on the reals indexed by elements of some set X. A **block** is a subset of $\mathbb{R}^X$ of the form $\prod \vec{B} = \{\vec{r} \in \mathbb{R}^X : \forall x \in X\ \vec{r}(x) \in B_x\}$, where $\vec{B} = \langle B_x : x \in X \rangle$ is a sequence whose coordinates are Borel sets of reals positive with respect to the corresponding ideal in the $\vec{I}$-sequence. $MRR(\vec{I})$ (for *mutually rectangularly Ramsey*) is the statement that the collection of Borel subsets of $\mathbb{R}^X$ containing no block as a subset, is a σ-ideal. If $MRR(\vec{I})$ holds then $\prod \vec{I}$ denotes this σ-ideal. If $\vec{I}$ is a sequence of arbitrary length, $MRR(\vec{I})$ is the statement that $MRR(\vec{J})$ holds for every countable subsequence $\vec{J}$ of $\vec{I}$.

Fact 1.1 is just the statement $MRR(J, J)$, where J is the σ-ideal of countable sets on the reals. Statements of the form $MRR(\vec{I})$ are usually proved using sophisticated methods in Ramsey theory such as Millikan's theorem. One point in this paper is that they can be frequently proved just from the topological forcing properties of the factor posets P_I for ideals I in the sequence $\vec{I}$, where

Definition 1.3: If I is a σ-ideal on the reals, the symbol P_I denotes the factor poset of Borel I-positive sets of reals ordered by inclusion. If $\vec{I}$ is a sequence of ideals then $P_{\vec{I}}$ is the side-by-side countable support product of the factor posets of σ-ideals in the sequence. Note that if the sequence $\vec{I}$ is countable and $MRR(\vec{I})$ holds, then $P_{\vec{I}}$ naturally densely embeds into the poset P_J where $J = \prod \vec{I}$.

The key forcing property in this context turns out to be

Definition 1.4: $\Phi(P)$ is the statement that P is proper, and every Borel meager set in the extension is covered by one coded in the ground model.

This is a rather traditional property known to be preserved under countable support iterations—[1], Theorem 6.3.22, or [9], Theorem 5.4.10. I will show that in the definable context it is preserved even under countable support side-by-side products:

THEOREM 1.5 (LC): *Suppose that $\vec{I}$ is a sequence of definable σ-ideals, and suppose that $\Phi(P_I)$ holds for every ideal I in it. Then $\Phi(P_{\vec{I}})$ and $MRR(\vec{I})$ hold.*

Here LC denotes the use of a suitable large cardinal axiom. The connection of this axiom with the notion of definability in the statement of the theorem will be explained at the end of this introduction.

The rectangular Ramsey property falls out from the argument as a consequence of the computation of the ideal associated with the product. A variation of the proof gives another preservation theorem. Consider the following property of partial orders:

Definition 1.6: $\Psi(P)$ is the statement that P is proper and every Borel Lebesgue null set in the extension is covered by one coded in the ground model.

This is another forcing property preserved by countable support iterations—[1], 6.3.F, or [9], 5.4.10. It implies Φ—[1], Theorem 2.3.1, and it turns out that it is very closely related to Φ.

THEOREM 1.7 (LC): *Suppose that $\vec{I}$ is a sequence of definable σ-ideals on the reals. If $\Psi(P_I)$ holds for every ideal I in it, then $MRR(\vec{I})$ and $\Psi(P_{\vec{I}})$ hold.*

A similar argument gives an asymmetric preservation result. Consider the following property of partial orders.

Definition 1.8: $\Theta(P)$ is the statement that P is proper and the set of ground model reals is not meager in the extension.

Again, this forcing property is preserved under definable countable support iterations. The preservation theorems in the above style fail for side-by-side products of definable posets satisfying Θ. However, the following is still true.

THEOREM 1.9 (LC): *Suppose that I, J are definable σ-ideals such that $\Theta(P_I)$ and $\Phi(P_J)$ hold. Then $MRR(I, J)$ and $\Theta(P_I \times P_J)$ hold.*

The theorems in this paper work only for ideals which are in a suitable sense definable. One way to make this precise is to demand that for the relevant ideals the collection of (codes for) analytic sets which belong to the ideal is projective. The theorems also require a suitable large cardinal axiom to guarantee the determinacy of the various infinite games with real entries used in the arguments. The existence of $\omega + \omega$ Woodin cardinals is sufficient by the results of the forthcoming [5]. Most definable ideals which occur in the forcing practice actually have much simpler definitions, and a brief survey of available examples will show that the winning strategies for the infinite games are readily at hand in all particular cases. However, the definition of the payoff sets of the games even in the syntactically simplest situations is complex enough to necessitate the large cardinal axioms for the general argument to go through. Thus the large cardinal hypothesis can be understood as a price for a very general theorem, and restricting attention to a smaller class of ideals does not seem to decrease that price.

The notation used in the paper sticks to the set theoretic standard of [2]. For a finite binary sequence s the term $[s]$ denotes the open set of all infinite binary sequences containing s as an initial segment.

2. Several examples

While there are not many definable partial orders P with $\Phi(P)$ that occurred in practice, they do not seem to admit a simple classification. This is partially due to the fact that the property is preserved by countable support iterations and products. The following examples are more or less well-known [9].

Example 2.1: Let I be the ideal of countable subsets of $\mathbb{R}$. The Sacks forcing can be densely embedded in the factor P_I, and Fact 1.1 just translates into the statement $MRR(I, I)$.

Example 2.2: Let E_0 be the modulo finite equality on 2^ω and let I be the σ-ideal generated by Borel sets which visit each E_0-equivalence class in at most one point. The rectangular Ramsey property $MRR(I, I, I, \ldots)$ is proved for example in [9] 2.3.16, which also gives a computation of a simple basis for the ideal $\prod(I, I, I, \ldots)$ in terms of the ideal I.

Example 2.3: Let G be the graph on 2^ω connecting two binary sequences just in case they differ in exactly one point. Let I be the σ-ideal generated by the Borel G-independent subsets of 2^ω. The Silver forcing densely embeds into P_I. The situation is parallel to that of E_0-forcing.

Example 2.4: Let $c_{\min}$ be the clopen partition of pairs of infinite binary sequences in two classes defined by $c_{\min}(x, y) = x\Delta y \bmod 2$, where $x\Delta y$ is the smallest number where the sequences x, y differ. Let I be the σ-ideal generated by $c_{\min}$ homogeneous sets. P_I is a forcing similar to Sacks forcing, and it satisfies $\Phi(P_I)$ as well.

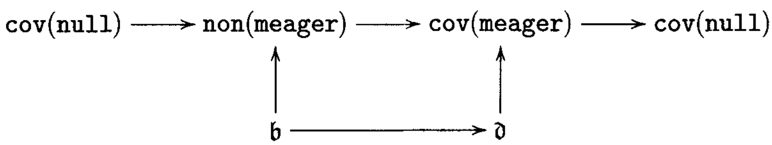
All of these forcings actually satisfy the stronger property Ψ. The simplest example of a partial order with $\Phi \wedge \neg\Psi$ is the following:

Example 2.5: Let $\{a_n : n \in \omega\}$ be a collection of nonempty finite pairwise disjoint subsets of ω, $|a_n| \geq n$. Let I be the ideal on 2^ω σ-generated by sets $B_f = \{g \in 2^\omega : \forall n\ f \restriction a_n \neq g \restriction a_n\}$ as f ranges over all infinite binary sequences. It is not difficult to see that every analytic I-positive set has a subset of the form $[T]$, where $T \subset 2^{<\omega}$ is a nonempty tree such that for every $t \in T$ there is a number $n \in \omega$ such that for every function $h \in 2^{a_n}$ there is a node $s \in T$ extending both t and h. It follows that the poset P_I satisfies the property Φ. However, it fails the property Ψ since if $g \in 2^\omega$ is the P_I-generic function, all ground model reals are included in the null set

$$\{f \in 2^\omega : \text{there are infinitely many } n \text{ such that } f \restriction n = g \restriction n\}$$

and so this set cannot be covered by a ground model coded null set.

The non-examples are perhaps more interesting than the examples. Exactly which instances of rectangular Ramsey properties does Theorem 1.5 not cover and why? Recall the relevant part of Cichoń's diagram:



It is well-known that cof(meager) is the maximum of $\mathfrak{d}$ and non(meager)—[1], Theorem 2.2.11. Thus forcings which fail Φ either must fail to be bounding or must make the ground model reals meager. What happens if the ideals in question violate the bounding condition?

Example 2.6: Let I be the σ-ideal generated by compact subsets of ω^ω. The Miller forcing densely embeds into the factor P_I. Spinas [7] proved that $P_I \times P_I$ is proper and in fact $MRR(I, I)$ holds. However, $MRR(I, I, I)$ fails rather badly by a result of [8]: there is a Borel map $f\colon (\omega^\omega)^3 \to \omega^\omega$ such that the image of every block with superperfect sides contains a nonempty open set. Then for every dense codense set $X \subset \omega^\omega$ the coloring $c\colon (\omega^\omega)^3 \to 2$ given by $c(x, y, z) = 0$ iff $f(x, y, z) \in X$ witnesses the failure of $MRR(I, I, I)$.

It is now in fact easy to observe that if $J_i : i \in 3$ are σ-ideals such that P_{J_i} are proper not bounding notions of forcing, then $\neg MRR(J_0, J_1, J_2)$. Fix a Borel coloring $c\colon (\omega^\omega)^3 \to 2$ with no monochromatic block with superperfect sides. The assumption on the ideals J_i implies that there are J_i-positive Borel sets B_i and Borel functions $f_i\colon B_i \to \omega^\omega$ such that for every J_i-positive Borel subset $C_i \subset B_i$ the set $f_i''C_i$ is not σ-compact. Let $d\colon \prod_{i\in 3} B_i \to 2$ be defined by $d(x, y, z) = c(f_0(x), f_1(y), f_2(z))$. It is clear from the construction that there is no monochromatic block with J_i-positive Borel sides.

Now what if the poset P_I violates the nonmeagerness condition?

Example 2.7: Consider the ideal I of Lebesgue null sets of the reals with its attendant Solovay forcing P_I. Arnold Miller pointed out to me the most elegant way to see that $MRR(I, I)$ fails. It is a well-known fact (Theorem 3.2.10 of [1]) that for every two Borel I-positive sets A, B the set $A + B$ contains an interval. Then argue just like in the previous example.

Looking back at Cichoń's diagram, it is clear that we are squeezed into a quite small space in the search for ideals I for which $MRR(I, I, I)$ holds but is not implied by Theorem 1.5.

Example 2.8: A typical bounding forcing making the ground model reals meager without adding random reals is the fat tree forcing. A fat tree is a tree $T \subset \omega^{<\omega}$ such that for every $n \in \omega$ there is $m \in \omega$ such that every node in the tree of length at least m has at least n many immediate successors. The fat tree forcing is then the partial order of all fat trees ordered by inclusion. To evaluate the attendant σ-ideal, note that the collection I of all Borel subsets of ω^ω containing no subset of the form $[T]$ for some fat tree T is a σ-ideal. I do not know whether $MRR(I,I)$, $MRR(I,I,I)$ hold.

The rectangular Ramsey properties concerning several different ideals are somewhat more complex.

Example 2.9: Let I be the Lebesgue null ideal on 2^ω and J the Laver ideal on ω^ω, generated by sets $A_g = \{f \in \omega^\omega\colon$ for infinitely many $n \in \omega$, $f(n) \in g(f \restriction n)\}$ as g varies through all functions from $\omega^{<\omega}$ to ω. The Laver forcing P_J is connected with the invariant $\mathfrak{b}$—see the above diagram and compare this case with the previous ones. $MRR(I,J)$ fails. Consider the Borel function $f\colon 2^\omega \times \omega^\omega \to 2^\omega$ given by $f(a,b) = a \circ b$. It turns out that the image of every Borel rectangle with positive sides contains a nonempty open subset of 2^ω. The argument is completed just as in the previous cases. A short inspection of the proof will show that whenever K_0, K_1 are σ-ideals such that both factor forcings P_{K_0}, P_{K_1} are proper and one of them adds a splitting real and the other a dominating real, $MRR(K_0,K_1)$ must fail. Note that $\mathtt{cov}(I) = \mathfrak{b} \leq \mathtt{cof(meager)}$.

Example 2.10: Let P be the eventually different real forcing [1], 7.4.B, and let I be its associated σ-ideal on ω^ω, so that P is in the forcing sense equivalent to P_I. Let J be the ideal of countable sets. $MRR(I,J)$ fails. The easiest way to see that is to choose a perfect subset $X \subset \omega^\omega$ consisting of mutually eventually different functions, and define a function $f\colon \omega^\omega \times X \to 2^\omega$ by letting $f(y,x)$ be the sequence of parities of elements of the set $\{n \in \omega\colon x(n) = y(n)\}$. It turns out that the f-image of every rectangle with Borel I-positive and J-positive sides, respectively, contains a nonempty open subset of 2^ω. Just like in the previous arguments this means that $MRR(I,J)$ fails and in fact $MRR(I,K)$ fails for every nonprincipal σ-ideal K. Note that $\mathtt{cov}(I) \leq \mathtt{non(meager)} \leq \mathtt{cof(meager)}$.

The forcing preservation theorems stated in the paper also bring up more general questions:

QUESTION 2.11: *Suppose that I, J are definable σ-ideals such that $\mathtt{cof(meager)}$ is less than both $\mathtt{cov}(I \restriction B)$ and $\mathtt{cov}(J \restriction C)$ for all Borel I-positive sets B and Borel J-positive sets C. Does $MRR(I,J)$ hold?*

QUESTION 2.12: *Are there (necessarily undefinable) σ-ideals I, J such that both $\Phi(P_I)$ and $\Phi(P_J)$ hold, but MRR(I, J) fails?*

3. The fusion games

Fix a definable σ-ideal I; for simplicity and without loss of generality assume that its underlying space is 2^ω. Consider the following infinite game $G_\Phi(I)$ of length ω with real entries, between players Oldřich and Božena [3]. All moves in it are (codes for) Borel I-positive sets.

First, Božena indicates an initial set B. The game then has infinitely many rounds. In round $i \in \omega$, Oldřich indicates finitely many sets $B(i,j)$: $j \in j(i)$, and Božena responds to each by playing its subset $C(i,j)$. The order of moves is $B(i,0), C(i,0), B(i,1), C(i,1), \ldots$. It is Oldřich who calls a stop to the round i at some stage $j(i)$. Oldřich wins if no round dragged on for infinitely many moves, and the set $X = B \cap \bigcap_{i\in\omega} \bigcup_{j\in j(i)} C(i,j)$ does not belong to the ideal I. The set X will be referred to as the **result** of the play of the game $G_\Phi(I)$.

Example 3.1: Oldřich has a winning strategy in the game $G_\Phi(I)$ where I denotes the σ-ideal of countable sets. During the play he will stop at the i-th round after 2^i many moves, construct binary sequences $s(i,j)$ such that $s(i+1, 2j)$ and $s(i+1, 2j+1)$ are incompatible extensions of $s(i,j)$ and the moves $B(i+1, 2j) \subset [s(i+1, 2j)]$ and $B(i+1, 2j+1) \subset [s(i+1, 2j+1)]$ will be perfect subsets of $C(i,j)$. In the end of the play, the resulting set will be equal to the perfect set $\{r \in 2^\omega : \forall i \in \omega\ \exists j \in 2^i\ s(i,j) \subset r\}$.

Example 3.2: Božena has a winning strategy in the game $G_\Phi(I)$ where I denotes the ideal of Lebesgue null sets. She just plays so that the Lebesgue measure of the set $C(i,j)$ is less than 2^{-ij}. Then clearly the result X of the play is Lebesgue null, and she wins.

Perhaps some remarks on the nature of the game are in order. It is clear that in a play of the game Oldřich attempts to produce some sort of a fusion sequence. It is in general impossible for him to predict how long the different rounds will take. This would lead to a game related to the property Ψ as opposed to Φ. Compare this to Example 2.5. It is also impossible for him to expose all his moves in a given round at the outset of the round. This would allow him to construct Borel positive sets of mutually generic reals, which cannot be done in the case of E_0 forcing or Silver forcing. It is clear that for both sides a smaller move is a better move, and so the moves can be restricted to an arbitrary dense subset of the poset P_I.

The following lemma records the key connection between the property Φ and the game G_Φ.

LEMMA 3.3 (LC): *$\Phi(P_I)$ iff Oldřich has a winning strategy in the game $G_\Phi(I)$.*

The right-to-left direction is easy. Fix a winning strategy σ for Oldřich. To prove the properness of the poset P_I, let M be a countable elementary submodel of some large structure containing σ and let $B \in P_I \cap M$ be a condition. To find a master condition for the model M below the set B, let $\{D_i : i \in \omega\}$ enumerate the open dense subsets of the poset P_I in the model M and simulate a play of the game $G(I)$ against the strategy σ in which Božena plays the initial set B and then chooses her moves $C(i,j)$ to come from the sets $M \cap D_i$. It is easy to argue inductively that this is possible and all the moves of the play will be in the model M. The result X of the play will be an I-positive Borel set below the condition B, and clearly the required master condition for the model M.

To see that every Borel meager set in the extension is covered by one coded in the ground model, let $B \in P_I$ be a condition and $\{\dot{O}_n : n \in \omega\}$ a name for a sequence of open dense subsets of $2^{<\omega}$. Fix a bijection $f : \omega \to 2^{<\omega} \times \omega$ and simulate a run of the game G_Φ in which Oldřich follows his strategy σ, Božena plays the initial set B and in round $i \in \omega$ plays sets $C(i,j)$ in such a way that there is a descending chain $\langle t(i,j) : j \leq j(i) \rangle$ of binary sequences such that writing $f(i) = \langle s, n \rangle$, the chain begins with $t(i,0) = s$ and $C(i,j) \Vdash \check{t}(i,j+1) \in \dot{O}_n$. This is easily possible. Clearly, the resulting set $X \subset B$ of the play will force that $t(i,j(i)) \in O_n$. Thus the set $N_n = \{t \in 2^{<\omega} : X \Vdash \check{t} \in \dot{O}_n\} \subset 2^{<\omega}$ is open dense for every number $n \in \omega$, and the condition X forces the meager set $\{r \in 2^\omega : \exists n\, \forall m\; r \restriction m \notin O_n\}$ in the extension to be a subset of the meager set $\{r \in 2^\omega : \exists n\, \forall m\; r \restriction m \notin N_n\}$ coded in the ground model.

The opposite direction of Lemma 3.3 is harder. The key tool is the reduction of the game $G_\Phi(I)$ to an integer game $H_\Phi(I, C)$, where C is a compact I-positive set. This is a game of length ω between players Přemysl and Libuše [3]. The game has infinitely many rounds. In each round i Přemysl produces a finite collection $k(i,j) : j \in j(i)$ of natural numbers, and Libuše answers each with a sequence $s(i,j) \in 2^{k(i,j)}$. The order of moves is $k(i,0), s(i,0), k(i,1), s(i,1), \ldots$. It is Libuše who decides to call a stop to the round i at some stage $j(i)$. Libuše wins if no round dragged on for infinitely many moves and the set $X = \{r \in C : \forall i \in \omega \exists j \in j(i)\; s(i,j) \subset r\}$ is I-positive. The set X will be referred to as the **result** of the play.

CLAIM 3.4 (LC): *$\Phi(I)$ implies that Libuše has a winning strategy in the game $H_\Phi(I, C)$ for every compact I-positive set C.*

Proof: The game is determined by our large cardinal assumptions. So it is enough to derive a contradiction from the assumption that Přemysl has a winning strategy σ for some game $H_\Phi(I, C)$.

First observe that in this case Přemysl has a *positional* winning strategy τ in the form of an increasing sequence $\langle k_l : l \in \omega \rangle$ of natural numbers such that if he plays them successively he wins no matter what Libuše's moves are. This follows from the fact that a larger number is a better move, as far as Přemysl is concerned. To define the numbers k_l note that at each move of the game $H(I, C)$ Libuše has just finitely many options at her disposal, and so for every number l there is just a finite set Z_l of numbers that the strategy σ can possibly use at the l-th move. Let then $k_l = \max(Z_l \cup \{k_{l-1}\}) + 1$. For every play x against the strategy τ with Libuše's moves enumerated as s_l consider the play y against the strategy σ in which Libuše calls stops to rounds at the same places as in the play x and she plays the sequences $s_l \restriction k$ where $k \in k_l$ is the number the strategy σ produces at the l-th move. Clearly, the result of the play x is a subset of the result of the play y, and since the strategy σ was winning, both of these resulting sets must belong to the ideal I. Thus the strategy τ is winning as well.

Now fix a positional winning strategy $\tau = \langle k_l : l \in \omega \rangle$ for Přemysl as in the previous paragraph. Let $r \in C$ be a V-generic real for the poset P_I below the condition C. Let $g \in \prod_l 2^{k_l}$ be a function in the extension defined by $g(l) = r \restriction k_l$. Since the ground model reals are not meager, there is a ground model function $h \in \prod_l 2^{k_l}$ such that the set $u = \{l \in \omega : h(l) = g(l)\}$ is infinite. Since the ground model reals are dominating, there is a ground model infinite set $v \subset \omega$ such that between every two successive elements of it there is an element of the set u. Now consider the play of the game $H(I, C)$ against the strategy τ in which Libuše plays the sequences $s_l = h(l)$ and calls stops to the rounds after each l-th move where $l \in v$. It follows from the definition of the resulting set X of the play and the choice of h and v that $r \in X$. However, this play is in the ground model, so its result X is an I-small closed set coded in the ground model, and the generic real r is forced to fall out of all such sets. Contradiction! ∎

For the reduction of the game G_Φ to the game H_Φ I will need a small general observation which does not concern these games.

CLAIM 3.5: *If the poset P_I is proper and bounding, then compact sets are dense in it. Moreover, for every countable elementary submodel M of a large structure and a condition $B \in M \cap P_I$, there is a compact I-positive set $C \subset B$*

such that for every dense set $D \subset P_I$ in the model M there is a number $k \in \omega$ such that for every sequence $s \in 2^k$ there is an element $E \in D \cap M$ such that $C \cap [s] \subset E$.

Proof: The first sentence is proved in [9]. For the rest of the claim, fix the model M and a condition $B \in M \cap P_I$. Enumerate the infinite maximal antichains in M consisting of compact sets by $\{A_n : n \in \omega\}$, and for each of them enumerate the set $A_n \cap M$ by $\{E(n,m) : m \in \omega\}$. Finally, choose a master condition $B' \subset B$ for the model M.

Note that $B' \Vdash \forall n\ \exists! m\ E(n,m) \in \dot{G}$, and let $\dot{f} \in \omega^\omega$ be the name for a function assigning to each number n the unique m such that $E(n,m) \in \dot{G}$. The forcing P_I is bounding, and so there is a ground model function $g \in \omega^\omega$ and a condition $B'' \subset B'$ such that $B'' \Vdash \dot{f} < \check{g}$ pointwise. Let $B''' = B' \cap \bigcap_{n\in\omega} \bigcup_{m\in g(n)} E(n,m)$. The set B''' is Borel, and since the condition B'' forces the generic real into it, it has to be I-positive. Let $C \subset B'''$ be any compact I-positive subset.

Now for every number $n \in \omega$ the finitely many compact sets

$$\{E(n,m) : m \in g(n)\}$$

in the antichain A_n cover the compact set C, and by a compactness argument there must be a number k such that for every sequence $s \in 2^k$ there is some $m \in g(n)$ such that $C \cap [s] \subset E(n,m)$. The claim immediately follows. ∎

Now assume that $\Phi(P_I)$ holds. Note that the large cardinal assumptions imply that the game $G_\Phi(I)$ is determined, and it is enough to obtain a contradiction from the assumption that Božena has a winning strategy σ in it. Let M be a countable elementary submodel of a large structure containing the strategy σ and let $B \in M \cap P_I$ be the initial move dictated by the strategy. Let $C \subset B$ be the compact set from the previous claim, and let τ be Libuše's winning strategy in the game $H(I,C)$ from Claim 3.4. The contradiction will be reached by pitting the strategies τ and σ against each other in a way.

Find plays x and y of the games $G_\Phi(I)$ and $H_\Phi(I,C)$ observing the strategies σ and τ respectively so that

- all moves of the play x are in the model M,
- Oldřich calls stops to rounds in the play x exactly when Libuše calls stops in the play y,
- at move l, writing C_l for the l-th Božena's move in the play x and s_l for the l-th Libuše's move in the play y it is the case that $C \cap [s_l] \subset C_l$.

After this is done, it is clear from the second and third items that the result of the play y is a subset of the result of the play x, and since Libuše's strategy σ was winning, it must be the case that both of the resulting sets must be I-positive. This means that Božena lost the play x observing her strategy σ, and this contradiction will finish the proof.

The plays x and y are built simultaneously by induction. Suppose that $l \in \omega$ is a number and the partial plays $x \restriction l$ and $y \restriction l$ have been built. The set $Z = \{E \in P_I$: there is $D \in P_I$ such that $x \restriction l^\frown D^\frown E$ is a play of the game $G_\Phi(I)$ observing the strategy $\sigma\}$ is dense in P_I and belongs to the model M. By the choice of the set C there is a number $k_l \in \omega$ such that for every sequence $s \in 2^{k_l}$ there is a set $E \in Z \cap M$ such that $C \cap [s] \subset E$. Put k_l to be the next Přemysl's move in the play y. The strategy τ responds with some sequence s_l. By the choice of the number k_l there are sets B_l and C_l in the model M such that the play $x \restriction (l+1) = x \restriction l^\frown B_l^\frown C_l$ respects the strategy σ and $C \cap [s_l] \subset C_l$. This completes the induction step and the proof of Lemma 3.3.

4. The products

Theorem 1.5 now follows by a rather standard, if notationally awkward, argument. Suppose that X is a countable set and $\vec{I} = \langle I_x \colon x \in X\rangle$ is a countable collection of definable σ-ideals such that $\Phi(I_x)$ holds for every $x \in X$. Consider a game G defined in the same way as G_Φ except that now the moves are blocks and Oldřich wins if the result of the game contains a block.

CLAIM 4.1: *Oldřich has a winning strategy in the game G.*

Proof: For the simplicity of notation I will deal with the case $X = 2$. The infinite case is very similar with only more confusing notation.

Consider the games $G_\Phi(I_0)$ and $G_\Phi(I_1)$. To facilitate the expressions that follow, index the moves in them in the order they come: B_0 (the initial Božena's move) $B_0(k) : k \in \omega$ (Oldřich's moves), $C_0(k)$: $k \in \omega$ (Božena's moves) in the game $G_\Phi(I_0)$ and $B_1, B_1(k), C_1(k)$: $k \in \omega$ in the game $G_\Phi(I_1)$. The moves in the game G will be denoted by $B_0 \times B_1$, $B_0(k) \times B_1(k), C_0(k) \times C_1(k)$: $k \in \omega$.

Use Lemma 3.3 to find Oldřich's winning strategies σ_b: $b \in 2$ for the respective games $G_\Phi(I_b)$. I will produce a winning strategy for Oldřich in the game G. The strategy is uniquely given by the following demands on the moves it produces:

- The play $p_0 = B_0, B_0(k), C_0(k)$: $k \in \omega$ follows the strategy σ_0. Let $0 = l_0 < l_1 < l_2 < \cdots$ be the natural numbers such that the strategy σ_0 calls stops to rounds in the play p_0 just before playing $B_{l_1}, B_{l_2}, \ldots$.

- For every number $k \notin \{l_i: i \in \omega\}$ the set $B_1(k)$ is just $C_1(k-1)$.
- For numbers $k \in \{l_i: i \in \omega\}$ the sets $B_1(k)$ are such that the play $p_1 = B_1, B_1(l_i), C_1(l_{i+1}-1): i \in \omega$ follows the strategy σ_1. Let $0 = m_0 < m_1 < m_2 < \cdots$ be the natural numbers such that the strategy σ_1 calls stops to rounds in the play p_1 just before playing $B_{m_1}, B_{m_2}, \ldots$. Thus $\{m_j: j \in \omega\} \subset \{l_i: i \in \omega\}$.
- Oldřich calls stops to rounds in the play of the game G just before the moves indexed by $m_j: j \in \omega$.

I must argue that in the end, Oldřich won the play of the game G. Let $D_b: b \in 2$ be the respective results of the plays p_b of the games $G_\Phi(I_b)$. Since these plays conformed to the winning strategies σ_b, the sets are I_b-positive respectively. It will be enough to show that the block $D_0 \times D_1$ is a subset of the result of the play of the game G. And indeed, suppose that $\langle r_0, r_1\rangle \in D_0 \times D_1$ is a pair. For every number $j \in \omega$ I must find some k, $m_j \leq k < m_{j+1}$ such that $r_0 \in C_0(k)$ and $r_1 \in C_1(k)$. To do this, first use the fact that r_1 belongs to the result of the play p_1 to find a number i such that $m_j < l_i \leq m_{j+1}$ and $r_1 \in C_1(l_i - 1)$. Then use the fact that r_0 belongs to the result of the play p_0 to find a number k such that $l_{i-1} \leq k < l_i$ and $r_0 \in C_0(k)$. By the second item above it is the case that $C_1(k) \supset C_1(l_i - 1)$ and so $r_1 \in C_1(k)$ as well. The proof is complete. ∎

Corollary 4.2: *The collection of Borel subsets of $(2^\omega)^X$ containing no block is a σ-ideal.*

Proof: Let $p_0 = \prod_{x\in X} B_x$ be a block, decomposed into a countable union $p_0 = \bigcup_m C_m$ of Borel sets. It is enough to show that one of the sets C_m contains a block.

Write $\vec{r}_{gen}$ for the $P_{\vec{I}}$-generic sequence of reals. Note that $p_0 \Vdash \vec{r}_{gen} \in \dot{p}_0$, since for every index $x \in X$ the n-th coordinate $\vec{r}_{gen}(x)$ is forced to be P_{I_x}-generic below the set B_x and therefore to belong to the set B_x in the extension. By an absoluteness argument, there is a block $p_1 \subset p_0$ which forces $\vec{r}_{gen} \in \dot{C}_m$ for some definite number $m \in \omega$. I claim that the set C_m contains a block.

To see this, let M be a countable elementary submodel of a large enough structure containing the condition p_1, the set C_m, as well as a winning strategy σ for Oldřich in the game G. Enumerate all open dense subsets of the poset $P_{\vec{I}}$ in the model M as $\{D_i : i \in \omega\}$ and simulate a run x of the game G against the strategy σ with the initial move p_1 and such that all its moves are in the model M and during the i-th round Božena plays only sets from the open dense set

D_i. The result of the game contains some block $p_2 \subset p_1$. I claim that $p_2 \subset C_m$; this will complete the proof.

Let $\vec{r}$ be a sequence from the block p_2. It is easy to check that the collection of all blocks in the model M containing the sequence $\vec{r}$ is a filter on the poset $P_{\vec{I}} \cap M$. By the simulation above, this filter is M-generic and $\vec{r}$ is its associated generic real. By the forcing theorem applied in the model M, $M[\vec{r}] \models \vec{r} \in C_m$, and by an absoluteness argument $\vec{r} \in C_m$. So $p_2 \subset C_m$ as desired. ∎

Corollary 4.3: *$\Phi(P_{\vec{I}})$ holds.*

Proof: Writing $J = \prod \vec{I}$ it is now clear that the poset $P_{\vec{I}}$ naturally densely embeds into P_J, the game G is just $G_\Phi(J)$ under another name, and Oldřich has a winning strategy in it. A reference to Lemma 3.3 concludes the argument. ∎

This completes the proof of Theorem 1.5. I will state two corollaries of independent interest. To facilitate the notation in the statement and proof, for a given index x let $\vec{I} \ominus x$ be the sequence $\vec{I}$ with the x-th entry removed. Clearly $P_{\vec{I}} = P_{\vec{I}\ominus x} \times P_{I_x}$. Let $\vec{r}_{gen}$ be the $P_{\vec{I}}$-generic sequence, and let $\vec{r}_{gen} \ominus x$ be just $\vec{r}_{gen}$ with its x-th entry removed, understood now as the $P_{\vec{I}\ominus x}$-generic sequence.

Corollary 4.4: *For every number $n \in \omega$, $P_{\vec{I}} \Vdash \vec{r}_{gen}(\check{x})$ belongs to no Borel $\dot{I}_x$-small set coded in the model $V[\vec{r}_{gen} \ominus \check{x}]$.*

Of course, the real $\vec{r}_{gen}(x)$ belongs to no Borel I_x-small set coded in V, since it is V-generic for the poset P_{I_x}. The point of the corollary is that the real falls out even from all I_x-small Borel sets coded in the larger model $V[\vec{r}_{gen} \ominus x]$.

Proof: Suppose p is a $(\vec{I} \ominus x)$-block, let $\dot{U}$ be a $P_{\vec{I}\ominus x}$-name for an I_x-small set, and let $B \in P_{I_x}$ be a Borel set. It will be enough to find a $(\vec{I} \ominus x)$-block $q \subset p$ and a Borel I_x-positive set $C \subset B$ such that $\langle q, C\rangle \Vdash \dot{r} \notin \dot{U}$ in the product $P_{\vec{I}\ominus x} \times P_{I_x}$, where $\dot{r}$ is the name for the P_{I_x}-generic real.

Thinning out the block p if necessary we may assume that there is a Borel set $D \subset p \times B$ such that all vertical sections of the set D are I_x-small and $p \Vdash \dot{U} \cap \dot{B}$ is the vertical section of the set $\dot{D}$ corresponding to the sequence $(\vec{r}_{gen} \ominus x) \in \dot{p}$. Theorem 1.5 now implies $MRR(\prod \vec{I} \ominus x, I_x)$, and so either the set D or its complement in $p \times B$ must contain a rectangle with positive sides. Well, it cannot be the set D since its vertical sections are I_x-small, so there must be a rectangle $q \times C \subset (p \times B) \setminus D$. It is immediate that q, C work as required. ∎

COROLLARY 4.5: *If every ideal on the sequence $\vec{I}$ is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ then so is the ideal $\prod \vec{I}$.*

Proof: It follows from Lemma C.0.9 of [9] that if J is a σ-ideal such that the factor poset P_J is proper and bounding, then J is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ iff the set of J-positive compact sets is analytic iff there is an analytic dense collection of J-positive compact sets. Now write $J = \prod \vec{I}$. Theorem 1.5 implies that the factor forcing is proper and bounding. If every ideal on the sequence $\vec{I}$ is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$, then clearly the collection of all blocks of the form $\prod \vec{C}$, where $\vec{C}$ is a sequence of compact sets positive with respect to the corresponding ideal on the sequence $\vec{I}$, is analytic and dense in P_J. The corollary follows. ∎

5. Cofinality of the null ideal

The proof of Theorem 1.7 is almost identical. It uses the following key combinatorial fact:

FACT 5.1 ([1], Section 2.3): $\Psi(P)$ is equivalent to properness of P together with the statement "for every ground model nondecreasing function $h \in \omega^\omega$ diverging to infinity and for every function $f \in \omega^\omega$ in the extension, there is a ground model function $g\colon \omega \to [\omega]^{<\omega}$ such that for every number n the set $g(n)$ has size at most $h(n)+1$ and contains the value $f(n)$".

The only change in the proofs is that I must devise new games $G_\Psi(I)$, $H_\Psi(I,C)$. The game $G_\Psi(I)$ is played exactly as $G_\Phi(I)$ except it is now Božena who decides the lengths of the rounds, and it is her responsibility to see to it that the lengths of the rounds are finite, never decrease and diverge to infinity. The change in the definition of the H game is the same. Similarly as in the Φ case, the following claims are crucial:

CLAIM 5.2 (LC): *$\Psi(P_I)$ if and only if Oldřich has a winning strategy in the game $G_\Psi(I)$.*

CLAIM 5.3 (LC): *$\Psi(P_I)$ implies that Libuše has a winning strategy in the game $H_\Psi(I,C)$, for every closed I-positive set C.*

The only real difference occurs in the proof of the latter claim:

Proof: The game is determined, and it will be enough to derive a contradiction from the assumption that Přemysl has a winning strategy σ. First, use a compactness argument to find a nondecreasing function $h \in \omega^\omega$ diverging to infinity such that in all plays in which Přemysl uses his strategy, the i-th round will have at least $h(i)+1$ many moves. As in Claim 3.4, it is also possible to find a positional strategy τ in the form of an increasing infinite sequence $\langle k_n : n \in \omega \rangle$ of natural numbers such that Přemysl wins if he plays an arbitrary increasing subsequence of it and lets the i-th round last $h(i)+1$ steps, disregarding Libuše's moves entirely.

Now let $r \in C$ be a V-generic real under the condition C, and let $e \in \prod_n 2^{k_n}$ be the function defined by $e(n) = r \restriction k_n$. Since $\Psi(P)$ implies $\Phi(P)$, the ground model reals are not meager, and there is a ground model function $d \in \prod_n 2^{k_n}$ such that the set $u = \{n \in \omega : e(n) = d(n)\}$ is infinite. Using Fact 5.1 it is not difficult to find a ground model function $g: \omega \to [\omega]^{<\omega}$ such that for every number $i \in \omega$ the set $g(i)$ has size $h(i)+1$, contains some element of the set u, and moreover $\max(g(i)) < \min(g(i+1))$.

Consider the play of the game $H_\Psi(I, C)$ in which Přemysl plays numbers from the set $\{k_n : n \in \bigcup \mathrm{rng}(g)\}$ in the increasing order and lets the i-th round last for $h(i)+1$ many moves, and Libuše answers with sequences $d(n) \in 2^{k_n} : n \in \bigcup \mathrm{rng}(g)$. The result X of this play contains the real r by the choice of the functions g and d. However, the play is in the ground model, therefore the set X is an I-small closed set coded in the ground model and the generic real r is forced to fall out of all such sets. Contradiction! ∎

6. Uniformity of the meager ideal

The proof of Theorem 1.9 depends on a fusion game characterization of the property $\Theta(P_I)$ for definable σ-ideals I. Fix a partition $\omega = \bigcup_i a_i$ of the natural numbers into infinite sets and consider the game $G_\Theta(I)$ between players Oldřich and Božena of length ω. All moves in it are I-positive Borel sets again. Božena starts out with an initial set B_{ini} and then at each round j Oldřich plays a set B_j, which Božena answers with its subset C_j. Oldřich wins if the *result* of the play, the set $B_{ini} \cap \bigcap_i \bigcup_{j \in a_i} C_j$, does not belong to the ideal I.

Lemma 6.1 (LC): *$\Theta(P_I)$ if and only if Oldřich has a winning strategy in the game $G_\Theta(I)$.*

Proof: The right-to-left direction is easy. Let σ be a winning strategy for Oldřich. The proof of properness is the same as in Lemma 3.3 and it is left to the reader. To see that the ground model reals are not meager in the extension, let $B \in P_I$ be a condition and $\{\dot{O}_n : n \in \omega\}$ a name for a sequence of open dense subsets of $2^{<\omega}$. Simulate a play of the game $G_\Theta(I)$ in which Božena plays the initial move $B = B_{ini}$ and then on the side constructs a chain $t_0 \subset t_1 \subset \cdots$ of finite binary sequences and plays so that for every integer $i \in \omega$ and every $j \in a_i$ the condition C_j forces the sequence $\check{t}_j$ into $\dot{O}_n$. Let $r = \bigcup_n t_n$. The result of the play is then a condition in the poset P_I which forces the ground model real $\check{r}$ to have an initial segment in every set $\dot{O}_n$: $n \in \omega$.

For the left-to-right direction note that the game is determined by the large cardinal assumptions, and it is enough to derive a contradiction from the assumption that Božena has a winning strategy σ. Towards the contradiction, choose a countable elementary submodel M of a large enough structure containing the strategy σ and let T be the tree of all partial plays of the game $G_\Theta(I)$ respecting the strategy σ in the model M in which Božena makes the last move. For a node $t \in T$ let $last(t)$ be this last Božena's move in the play t.

It is clear that for every node $t \in T$ the set

$$D_t = \{last(s) \colon s \in T \text{ is an immediate successor of the node } t\}$$

is the intersection of some dense set $E_t \subset P$ in the model M with the model M itself, namely of the dense set

$$E_t = \{C \in P_I \colon \exists B \in P_I \; t^\frown B^\frown C \text{ respects the strategy } \sigma\}.$$

Thus, if $B_{ini} \in M$ is the initial move dictated by the strategy σ and $p \leq B_{ini}$ is some M-master condition below it, it is the case that $p \Vdash \forall t \in T \; \exists C \in D_t \; \dot{r}_{gen} \in C$. Let r be a generic real below the condition p. Since the set of all ground model branches of the tree T is not meager, there is a ground model branch $b \subset T$ such that for every number $i \in \omega$ there is $j \in a_i$ such that $r \in C_j$, where C_j is the set the strategy σ played on the j-th round of the round b.

Now the play $b \subset T$ is in the ground model, and its resulting set is a ground model coded I-small Borel set. The real r belongs to the resulting set by the choice of the play b, but at the same time it is forced to fall out of all such sets. Contradiction! ∎

There is an important corollary which greatly simplifies certain statements in [9]. Recall:

Definition 6.2 ([9], Chapter 4): A forcing P is **strongly proper** if for every countable elementary submodel M of a large enough structure, every condition $p_0 \in P \cap M$ and every collection $\{D_i : i \in \omega\}$ of dense subsets of the poset $P \cap M$ there is a *strong master* condition $p_1 \leq p_0$ forcing the generic filter to meet all the sets in the collection. Note that if $P = P_I$ for some σ-ideal I then this is equivalent to saying that the set $p_1 = p_0 \cap \bigcap_i \bigcup D_i$ is I-positive.

It turns out that this notion is in the definable context identical to Θ.

COROLLARY 6.3 (LC): *Let I be a definable σ-ideal. P_I is strongly proper if and only if $\Theta(P_I)$ holds.*

Proof: First suppose that $\Theta(P_I)$ holds, and use Claim 6.1 to find a winning strategy σ for Oldřich in the game $G_\Theta(P_I)$. Now suppose that M is a countable elementary submodel of a large enough structure containing I and σ, $B \in M \cap P_I$ is a condition, and $\{D_i : i \in \omega\}$ is a collection of dense subsets of the poset $P_I \cap M$. Simulate a play of the game $G_\Theta(I)$ in which Oldřich follows the strategy σ and Božena plays the set B as her initial move, and makes sure that for every number $i \in \omega$ and every $j \in a_i$, $C_j \in D_i$. The resulting set of the play will be the desired strong master condition.

On the other hand, suppose that a poset P is strongly proper. To show that the ground model reals are not meager, just choose a condition $p_0 \in P$ and a name $\{\dot{O}_i : i \in \omega\}$ for a collection of open dense subsets of $2^{<\omega}$. Let M be a countable elementary submodel of a large enough structure containing all the relevant objects. Using the fact that the set $P \cap M$ is countable, it is easy to inductively construct a sequence $t_0 \subset t_1 \subset \cdots$ of finite binary sequences such that for all $i \in \omega$ and $p \in P \cap M$ there is some $j \in \omega$ and a condition $q \in M$, $q \leq p$ such that it forces $\check{t}_j \in \dot{O}_i$. This is to say that the sets $D_i = \{q \in P \cap M : \exists j\ q \Vdash t_j \in \dot{O}_i\}$ are dense in the poset $P \cap M$ for all numbers $i \in \omega$. Let $p_1 \leq p_0$ be the strong master condition, and let $r = \bigcup_j t_j$. Clearly p_1 forces the real $\check{r}$ to have an initial segment in each of the sets $\dot{O}_n$ as desired. ∎

Towards the proof of Theorem 1.9, let I, J be definable σ-ideals, and suppose that $\Theta(I)$ and $\Phi(J)$ hold.

COROLLARY 6.4: *$\Theta(P_I \times P_J)$ holds.*

Proof: It is enough to show that the poset is strongly proper. Fix a winning strategy σ for Oldřich in the game $G_\Phi(J)$, let M be a countable elementary submodel of a large enough structure, $p_0 \times q_0 \in M$ a Borel $I \times J$ block in it, and $\{D_i : i \in \omega\}$ a collection of open dense subsets of the poset $P_I \times P_J \cap M$.

First note that there is a J-positive Borel set $q_1 \subset q_0$ such that for every number $i \in \omega$ the set $E_i = \{p \in P_I \cap M : \exists q \in M\ q_1 \leq q \wedge \langle p, q\rangle \in D_i\}$ is open dense in $P_I \cap M$. To see this, simulate a play of the game $G_\Phi(J)$ against the strategy σ in the same way as in the first two paragraphs of the proof of Lemma 3.3, with the poset $2^{<\omega}$ replaced by $P_I \cap M$.

Now since the forcing P_I is strongly proper, the set $p_1 = p_0 \cap \bigcap_i \bigcup E_i$ is Borel and I-positive. It is not difficult to see that $p_1 \times q_1 \subset p_0 \times q_0 \cap \bigcap_i \bigcup D_i$, and the block $p_1 \times q_1$ is the desired strong master condition. Thus the poset $P_I \times P_J$ is strongly proper. ∎

COROLLARY 6.5: *MRR(I, J) holds.*

Proof: Suppose that $p_0 \times q_0$ is an $I \times J$-block, decomposed into a countable union $\bigcup_m C_m$ of Borel sets. It is enough to show that one of the sets $C_m : m \in \omega$ contains a block. Since $p_0 \times q_0$ forces the generic pair of reals into itself, there must be a strengthening $p_1 \times q_1$ which forces the generic pair into a set C_m for some specific number $m \in \omega$. Let M be a countable elementary submodel of a large enough structure containing the sets p_1, q_1, C_m. The argument from the previous proof produces a block $p_2 \times q_2 \subset p_1 \times q_1$ consisting only of pairs of mutually M-generic reals. By the forcing theorem and an absoluteness argument, $p_2 \times q_2 \subset C_m$ as desired. ∎

Theorem 1.9 follows. To conclude the paper, I will restate and reprove a result of Shelah [6]:

COROLLARY 6.6 (LC): *Suppose that I is a definable c.c.c. σ-ideal. Exactly one of the following holds:*

- *P_I is in the forcing sense equivalent to the Cohen forcing;*
- *some condition in P_I forces the set of the ground model reals to be meager.*

It turns out that the same dichotomy holds true on the random side too. Under the assumption that I is a definable c.c.c. σ-ideal, either P_I is in the forcing sense equivalent to the Solovay forcing, or some condition in P_I forces the set of the ground model to be null. This result will appear in an unrelated forthcoming work.

Proof: It is well-known that in the Cohen extension, the set of ground model reals remains non-meager, and therefore the two options exclude each other. Now suppose the second item fails. Let M be a countable elementary submodel of a large enough structure and consider the forcing $Q = P_I \cap M$. The P_I-name $\dot{r}_{gen}$ for the generic real remains a $P_I \cap M$-name for a real. I claim that $Q \Vdash \dot{r}_{gen}$ falls out of all ground model coded Borel I-small sets.

For suppose this is not the case. Then there must be an I-small Borel set C and a condition $B_0 \in Q$ such that $B_0 \Vdash_Q \dot{r}_{gen} \in \dot{C}$. By the Baire category theorem applied to the space of all ultrafilters on the poset Q, there must be open dense subsets $\{D_i : i \in \omega\}$ of Q such that for every ultrafilter g on Q containing the condition B_0 and meeting all these open dense sets, it is the case that $\dot{r}_{gen}/g \in C$. Extending the collection of the open dense sets if necessary I may assume that it includes all the sets of the form $D \cap M$ where $D \subset P_I$ is an open dense set in the model M. Since the poset P_I is strongly proper, the set $B_1 = B_0 \cap \bigcap_i \bigcup D_i$ is Borel and I-positive. Every real from the nonempty set $B_1 \setminus C$ then generates an ultrafilter on the poset Q with properties contradicting the choice of the sets $\{D_i : i \in \omega\}$.

But now notice that since the σ-ideal I is assumed to be c.c.c., every real which falls out of all ground model coded I-small sets is P_I-generic. Therefore, the forcing Q adds a P_I-generic real, and so P_I is (at least locally) regularly embeddable into the complete algebra generated by the poset Q. However, the poset Q is countable, so in the forcing sense isomorphic to the Cohen forcing, and all complete subalgebras of the Cohen algebra are Cohen themselves! ∎

References

[1] T. Bartoszynski and H. Judah, *Set Theory. On the structure of the real line*, A K Peters, Wellesley, MA, 1995.

[2] T. Jech, *Set Theory*, Academic Press, San Diego, 1978.

[3] A. Jirásek, *Staré pověsti české*, Nákladem Josefa Vilímka, Praha, 1921.

[4] A. S. Kechris, *Classical Descriptive Set Theory*, Springer-Verlag, New York, 1994.

[5] I. Neeman, A book in preparation.

[6] S. Shelah, *Properness without elementaricity*, Journal of Applied Analysis **10** (2004), 168–289.

[7] O. Spinas, *Ramsey and freeness properties of Polish planes*, Proceedings of the London Mathematical Society **82** (2001), 31–63.

[8] B. Velickovic and W. H. Woodin, *Complexity of the reals in inner models of set theory*, Annals of Pure and Applied Logic **92** (1998), 283–295.

[9] J. Zapletal, *Descriptive Set Theory and Definable Forcing*, Memoirs of the American Mathematical Society, American Mathematical Society, Providence, RI, 2004.